# MATH GRE PREP: WEEK 5

UCHICAGO REU 2018

**Directions:** Each of the following 22 questions or incomplete statements below is followed by five suggested answers or completions. In each case, select the one that is best. In this test:

(1) All logarithms with an unspecified base are natural logarithms, that is, with base $e$;
(2) The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ denote the set of positive integers, integers, rational numbers, real numbers, and complex numbers, respectively;
(3) Any partial inverse function (e.g., $\arcsin = \sin^{-1}$) is assumed to take values on the principal branch cut (e.g., $[-\pi/2, \pi/2]$);
(4) If unspecified, a derivative is taken against the variable $x$;
(5) All integrals should be assumed to be (possibly improper) Riemann(-Stieltjes) integrals.

**(1)** What is the length of the curve determined by $x(t) = 4\sin(t/4)$ and $y(t) = 1 - 2\cos^2(t/4)$?

(A) $4(\sqrt{2} + 2\,\mathrm{arccosh}(1))$

(B) $4(\sqrt{2} + \mathrm{arcsinh}(1))$

(C) $8(\sqrt{2} + \mathrm{arccosh}(1))$

(D) $8(\sqrt{2} + \mathrm{arcsinh}(1))$

(E) $8(\sqrt{2} + 2\,\mathrm{arcsinh}(1))$

**(2)** Let $X = \mathbb{Z}_{>0}$. Define the Hjalmar Ekdal topology $\mathcal{T}$ on $X$ by $Y \in \mathcal{T}$ if the successor of every odd integer in $Y$ is also in $Y$. Which of the following are properties of this topology?

I. It is compact.

II. It is locally path connected. Namely, for every $x \in X$ and every neighborhood $N$ of $x$, there is a subneighborhood that is path connected.

III. It is totally disconnected, i.e. all connected components are points.

(A) None

(B) I and II only

(C) I and III only

(D) II only

(E) III only

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**(3)** Given the following system of equations, what is $x$?

$$\begin{aligned} x+y+z&=10\\ w+y+z&=7\\ w+x+z&=-3\\ w+x+y&=4 \end{aligned}$$

(A) $-2$

(B) $-1$

(C) 0

(D) 1

(E) 2

**(4)** Let $A, B, C$ be sets with $|A| = 15, |B| = 10$, which of the following ensure that $|C| \geq 25$?

I. There are surjections $C \to A$ and $C \to B$ and $A$ and $B$ are disjoint.

II. $A \cup B \subseteq C$.

III. $C \setminus B \subseteq A$.

IV. $\mathcal{P}(A \setminus B) \setminus C = \emptyset$.

(A) I only

(B) II only

(C) IV only

(D) II and III

(E) III and IV

**(5)** Let $A \subset \mathbb{R}$ be a set that contains the rationals.

I. If $A$ has positive measure (e.g., length), then $A = \mathbb{R}$.

II. If $A$ is open, then $A = \mathbb{R}$.

III. If $A$ is connected, then $A = \mathbb{R}$.

Which of the above must be true?

(A) All of these are true.

(B) II only

(C) I and III only

(D) III only

(E) I and II only

**(6)** What is

$$\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \cdots ?$$

(A) $\log 3$

(B) $\log 2$

(C) $2 \log 2$

(D) $e$

(E) $-\frac{1}{2} + \log 2$

**(7)** Let $\alpha, \beta \in \mathbb{R}$ be such that

$$\lim_{x \to 0} \frac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1.$$

What is $6(\alpha + \beta)$?

(A) 5

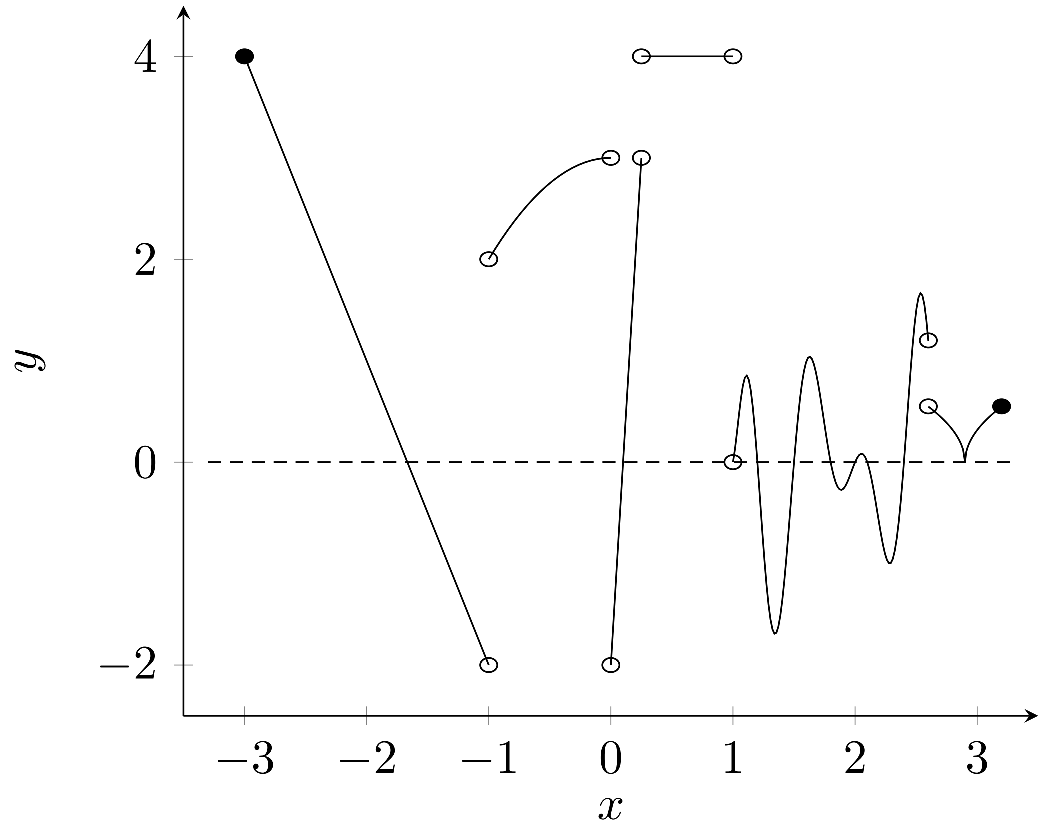
(B) 6

(C) 7

(D) 8

(E) 9

**(8)** Consider the following graph of $f''(x)$, for $f$ a function defined on $[-3, 3]$.




Which of the following is incorrect? (For ease of interpretation, note that every point given as an $x$-coordinate is a zero of $f''(x)$, e.g., the point $x = 0.1$ corresponds to the point $(0.1, 0)$ on the graph.)

(A) It is possible that $f$ is continuously differentiable.

(B) The function $f'$ achieves local maxima at $x = -5/3$, $x = 1.2$, $x = 1.8$, and $x = 2.1$.

(C) The function $f'''$ has a local maximum at $x = 0.1$.

(D) If $f'(x) = \int_0^x f''(x)dx$, then $f$ achieves its minimum in the range $[0, 0.25]$.

(E) The function $f''$ is differentiable wherever it is defined.

**(9)** Which of the following is the smallest value of $n$ for which the following limit exists for all $r \geq n$?

$$\lim_{(x,y)\to(0,0)} \frac{x^r}{|x|^2 + |y|^2}$$

(A) 1

(B) 1.5

(C) 2

(D) 2.5

(E) 3

**(10)** What is the length of the curve $\langle t, t \cdot \sin t, t \cdot \cos t \rangle$, $0 \leq t \leq \pi$?

(A) $\frac{\pi}{4}\sqrt{2+\pi^2} + \frac{1}{2}\operatorname{arcsinh}(\pi/\sqrt{2})$

(B) $\frac{\pi}{2}\sqrt{2+\pi^2} + \operatorname{arcsinh}(\pi/\sqrt{2})$

(C) $\pi\sqrt{2+\pi^2} + \operatorname{arcsinh}(\pi/\sqrt{2})$

(D) $\pi\sqrt{2+\pi^2} + 2\operatorname{arcsinh}(\pi/\sqrt{2})$

(E) $2\pi\sqrt{2+\pi^2} + 4\operatorname{arcsinh}(\pi/\sqrt{2})$

**(11)** Solve the following differential equation.

$$y' = \cos(x - y)$$

(A) $y - \tan\frac{x-y}{2} = C$

(B) $x + \tan\frac{x-y}{2} = C$

(C) $x + \cot(x - y) = C$

(D) $y + \sin(x - y) = C$

(E) $x + \cot\frac{x-y}{2} = C$

**(12)** For $r \in \mathbb{R}$, consider the limit:

$$\lim_{z \to e^{i\pi/2r}} \frac{z - e^{i\pi/2r}}{z^{2r}+1}.$$

What is the largest set (ordered by containment) where the above limit exists and is non-zero for all $r$ in the set?

(A) $\varnothing$

(B) $r > 0$, and $r$ is an integer

(C) $r \geq 1/2$

(D) $r > 0$

(E) $r \neq 0$

**(13)** Consider the matrix

$$\begin{pmatrix} 1 & 2 & x \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}.$$

For which value of $x$ is this matrix not invertible?

(A) $-1$

(B) 0

(C) 2

(D) 3

(E) 5

**(14)** Let $A$ be the unit 2-sphere in $\mathbb{R}^3$. Let $F = (x^3 - y^2z^4, 2y^3, z^3 - 3zy^2)$ be a vector field. Let $\overrightarrow{n}$ be the outward-pointing normal. Evaluate:

$$\iint_A F \cdot \overrightarrow{n}\, dS.$$

(A) $3\pi$

(B) $\pi$

(C) $\dfrac{12\pi}{5}$

(D) $\dfrac{-3\pi}{2}$

(E) 0

**(15)** A function $f$ is called Hölder continuous of exponent $\alpha$ if:

$$\exists c \in \mathbb{R} \;:\; \forall x, y,\ |f(x) - f(y)| \leq c|x - y|^{\alpha}.$$

Which of the following is incorrect?

(A) If $f\colon [0,1] \to \mathbb{R}$ is Hölder continuous of exponent $3/2$, then $f$ is constant.

(B) If $f\colon [0,1] \to \mathbb{R}$ is $C^1$, then $f$ is Hölder continuous of exponent $\alpha$, for all $0 \leq \alpha \leq 1$.

(C) If $f$ is Hölder continuous of exponent $\alpha > 0$, then $f$ is uniformly continuous.

(D) If $f\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, then $f$ is Hölder continuous of exponent $\alpha$ for all $0 < \alpha \leq 1$.

(E) The function $f(x) = \sqrt{x}$ is Hölder continuous of exponent $1/2$.

**(16)** Suppose that $A$ is a real matrix with non-negative eigenvalues, and $B$ is a real matrix with eigenvalues of absolute value less than one.

I. $I + A$ is invertible

II. $I + B$ is invertible

III. $I - A$ is inverible

IV. $I - B$ is invertible

Which of the above are true?

(A) II only

(B) II and IV only

(C) I and III only

(D) I, II, and IV only

(E) All of the above are true.

**(17)** What is the length of the curve $x^{2/3} + y^{2/3} = 4$?

(A) 48

(B) 52

(C) 56

(D) 60

(E) 64

**(18)** Suppose $A$ is a $3 \times 3$ matrix wtih entries in $\mathbb{R}$. Assume that $\det(A) = 6$, $\operatorname{tr}(A) = 6$, and that 3 is an eigenvalue of $A$. Compute $\operatorname{tr}(A^2)$.

(A) $-7$

(B) 7

(C) 14

(D) 36

(E) 40

**(19)** Let $M_n$ be the vector space of $n \times n$ matrices over $\mathbb{R}$. For a matrix $A \in M_n$, define $L_A \colon M_n \to M_n$ by

$$L_A(B) = AB.$$

Let $U$ be the subset of $M_n$ comprising of upper triangular matrices with diagonal entries 1 endowed with the obvious linear structure. Which of the following is false?

(A) The map $L_A \colon M_n \to M_n$ is linear.

(B) If $A \in U$ then the restriction $L_A |_U$ is a linear isomorphism of $U$.

(C) $\dim M_n = n^2$ and $\dim U = \frac{n(n-1)}{2}$

(D) If $A = \lambda I$ and $L_A \colon M_n \to M_n$ then $\det L_A = \lambda^{n^2}$.

(E) $L_A$ is invertible if and only if $A$ is invertible.

**(20)** Consider the polynomial

$$x^3 - 3x + a.$$

Which is the largest range of $a$ for which this polynomial has three distinct real roots?

(A) $a > 0$

(B) $|a| < 2$

(C) $|a| \leq 2$

(D) $|a| < 1/2$

(E) $|a| \leq 3$

**(21)** Find the maximum of $x^2y$ on the curve $x^2 + 2y^2 = 6$.

(A) 3

(B) 4

(C) 5

(D) 6

(E) 7

**(22)** Suppose that $N$ is a nonzero 2 by 2 matrix over $\mathbb{C}^2$ such that $N^{2018} = 0$. Then which matrix need $N$ be similar to, over $\mathbb{C}$, of course.

I.
$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

II.
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

III.
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

(A) I only

(B) II only

(C) I and II only

(D) I and III only

(E) I, II, and III

## Answers

**(1)** (B): Use definition of arc length. For the integral of $\sqrt{1+u^2}$, use hyperbolic substitution.

**(2)** (D): Obviously non-compact, and $\{1,2\}$ is a connected component. Note $f(x) = \lfloor x \rfloor$ is a path from $0 \to 1$.

**(3)** (B): Solve it (note: fastest to sum all of the equations).

**(4)** (C): Consider disjointness in the other cases.

**(5)** (D): Enumerate the rationals, and take exponentially decreasing open sets containing each one. Consider the union.

**(6)** (B): Recognize the Taylor series for log.

**(7)** (C): Use Taylor series to evaluate, should get $\alpha = 1$ and $\beta = 1/6$.

**(8)** (E): It is not differentiable at about $x = 2.8$.

**(9)** (D): You can take two different paths when $r = 2$; else, just let $y = 0$ to bound it from above.

**(10)** (B): Compute the integral (hyperbolic substitution).

**(11)** (E): Substitute $u = x - y$, solve separable differential equation.

**(12)** (C): There is an issue with choice of branch, e.g., let $r = 1/4$ and note $(e^{i\pi/2r})^{2r} = 1$.

**(13)** (D): Set determinant equal to zero.

**(14)** (C): Use Stokes theorem (after converting to spherical coordinates).

**(15)** (D): This is only true locally. Note the identity function is Lipschitz, but not Hölder.

**(16)** (D): Use Jordan blocks. Or recall that $\det(A+B) \geq \det(A) + \det(B)$.

**(17)** (A): This is an astroid; you could parameterize, to make the integral easier.

**(18)** (C): Eigenvalues are $1, 2, 3$; note eigenvalues of square is square of eigenvalues.

**(19)** (B): $U$ is not a linear subspace.

**(20)** (B): Differentiate to determine where local maximum/minimum is.

**(21)** (B): Use Lagrange multiplers.

**(22)** (E): Use Jordan Canonical form, and deduce.